Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 6, pp. 788–793, 2004

MATHEMATICAL MODELING OF TWO-PHASE CONVECTIVE FLOWS WITH FINE PARTICLES

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UDC 532.529.5

A two-phase convective flow with convection caused by a nonuniform distribution of solid particles is considered. The use of the mathematical model proposed is illustrated by an example of a two-phase flow in a shutter sedimentation reservoir.

Key words: *convective flow, two-phase medium, particle concentration.*

Introduction. In the present study, we model a two-phase flow whose dispersion phase is a rarefied "cloud" of fine solid particles. The term "fine particles" is used for particles whose velocity-stabilization (or relaxation) time during the free fall of particles in the liquid of interest is negligibly short compared to the characteristic sedimentation time of such particles in the computational domain.

We write the gravitational sedimentation time of a small solid particle as

$$\frac{d\boldsymbol{u}_s}{dt} = \frac{C_R}{m}(\boldsymbol{u} - \boldsymbol{u}_s) + \boldsymbol{g}\Big(1 - \frac{\rho_l^0}{\rho_s^0}\Big), \tag{1}$$

where $\boldsymbol{u}_s$ and ρ_s^0 are the velocity and density of the particle, $\boldsymbol{u}$ and ρ_l^0 are the velocity and density of the liquid, t is the time, m is the particle mass, and C_R is the drag coefficient of the particle. The drag C_R is generally a variable quantity: $C_R = C_D\rho_l^0 s|\boldsymbol{u} - \boldsymbol{u}_s|/2$, where $C_D = C_D(\mathrm{Re})$ is the drag coefficient that depends on the Reynolds number, s is the mid-section area of the particle, and Re is the Reynolds number.

We introduce the following notation: relaxation time $\tau = m/C_R$ and buoyancy force $\boldsymbol{F} = \boldsymbol{g}(1 - \rho_l^0/\rho_s^0)$. In the adopted notation, Eq. (1) acquires the form

$$\frac{d\boldsymbol{u}_s}{dt} + \frac{1}{\tau}\,\boldsymbol{u}_s = \frac{1}{\tau}\,\boldsymbol{u} + \boldsymbol{F}.$$

From here, the sought function can be expressed analytically as

$$\boldsymbol{u}_s = \exp\Big(-\int_0^t \frac{dt}{\tau}\Big)\Big[\boldsymbol{u}_{s0} + \int_0^t \Big(\frac{1}{\tau}\,\boldsymbol{u} + \boldsymbol{F}\Big)\exp\Big(\int_0^t \frac{dt}{\tau}\Big)\,dt\Big].$$

We integrate this expression by parts and obtain the following asymptotic expansion with respect to τ. We restrict ourselves to second-order terms with respect to τ and obtain

$$\boldsymbol{u}_s = \boldsymbol{u} + \tau\boldsymbol{F} - \tau\,\frac{d\boldsymbol{u}}{dt} + \exp\Big(-\int_0^t \frac{dt}{\tau}\Big)\Big[\boldsymbol{u}_{s0} - (\boldsymbol{u} + \tau\boldsymbol{F})_0 + \tau\,\frac{d\boldsymbol{u}}{dt}\Big|_0 + \theta(\tau^2)\Big] + \theta(\tau^2). \tag{2}$$

The approximate approaches in which the relaxation time of particles is assumed to be short are presently used in mathematical models of deep convection intended for local weather forecast. The system of deep convection

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Tomsk State University, Tomsk 634050; dekanat@ftf.tsu.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 6, pp. 19–25, November–December, 2004. Original article submitted December 25, 2003; revision submitted April 13, 2004.

equations was first proposed by Ogura and Phillips [1]. These authors, and also subsequent authors who used these equations (see [2–4]) did not discuss the applicability limits of this model. Meanwhile, two-phase flows contaminated by fine particles are widely used in many applications. Therefore, it makes sense to consider the class of convective flows with fine particles individually and discuss simplifications and assumptions that can be used to formulate adequate mathematical models.

Mathematical Model for Two-Phase Convection. Let us turn to constructing the model. Note that the derivative with respect to time in formula (2) is taken along the particle trajectory, i.e., $d\boldsymbol{u}/dt = \partial\boldsymbol{u}/\partial t + \boldsymbol{u}_s\nabla\boldsymbol{u}$. Next, we assume that the relaxation time τ is sufficiently short, and the terms of second-order smallness with respect to τ in (2) can be omitted. After the relaxation time, we may also neglect the influence of initial conditions. After some simple manipulations with the remaining terms in (2) we obtain

$$m\Big(\frac{\partial\boldsymbol{u}}{\partial t} + [\boldsymbol{u} + (\boldsymbol{u}_s - \boldsymbol{u})]\cdot\nabla\boldsymbol{u}\Big) \approx \frac{1}{2}C_D(\mathrm{Re})sg|\boldsymbol{u} - \boldsymbol{u}_s|(\boldsymbol{u} - \boldsymbol{u}_s) + m\boldsymbol{g}\Big(1 - \frac{\rho_l^0}{\rho_s^0}\Big).$$

It can be easily seen that, for a given flow-velocity field, one can find all projections of the particle-velocity lag vector $(\boldsymbol{u} - \boldsymbol{u}_s)$ by solving a system of algebraic equations.

To simplify the consideration, we assume that the two-phase convective flow is an incompressible laminar flow, two-dimensional in the plane (x, z) with velocities $\boldsymbol{u} = (u, v)$ and $\boldsymbol{u}_s = (u_s, v_s)$. We direct the z axis upward in the direction opposite to the gravity force. With the adopted assumptions, we write the projections of the equation of motion for the two-phase mixture onto the x and z axes:

$$\rho_l\Big(\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\nabla u\Big) + \rho_s\Big(\frac{\partial u_s}{\partial t} + \boldsymbol{u}_s\cdot\nabla u_s\Big) + \frac{\partial p}{\partial x} = \mu\Delta u,$$

$$\rho_l\Big(\frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\nabla v\Big) + \rho_s\Big(\frac{\partial v_s}{\partial t} + \boldsymbol{u}_s\cdot\nabla v_s\Big) + \frac{\partial p}{\partial z} = \mu\Delta v - g(\rho_l + \rho_s). \tag{3}$$

Here, $\rho_l = \rho_l^0(1 - \rho_s/\rho_s^0)$ is the mass of the liquid in a unit volume, ρ_s is the mass of the particles in a unit volume, p is the pressure, and μ is the viscosity of the liquid.

We introduce new variables $\delta u_s = u_s - u$ and $\delta v_s = v_s - v$, and also the vector $\boldsymbol{w} = (\delta u_s, \delta v_s)$. Next, we rewrite system (3) in terms of the new variables:

$$(\rho_l + \rho_s)\Big(\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\nabla u\Big) + \rho_s\Big(\frac{\partial\delta u_s}{\partial t} + \boldsymbol{w}_s\cdot\nabla u + \boldsymbol{u}_s\cdot\nabla\delta u_s\Big) + \frac{\partial p}{\partial x} = \mu\Delta u,$$

$$(\rho_l + \rho_s)\Big(\frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\nabla v\Big) + \rho_s\Big(\frac{\partial\delta v_s}{\partial t} + \boldsymbol{w}_s\cdot\nabla v + \boldsymbol{u}_s\cdot\nabla\delta v_s\Big) + \frac{\partial p}{\partial x} = \mu\Delta v - g(\rho_l + \rho_s). \tag{4}$$

Assuming that

$$\bar{u} = \frac{u}{V},\quad \bar{v} = \frac{v}{V},\quad \bar{u}_s = \frac{u_s}{V},\quad \bar{v}_s = \frac{v_s}{V},\quad \bar{\boldsymbol{u}} = \Big(\frac{u}{V}, \frac{v}{V}\Big),\quad \bar{\boldsymbol{w}}_s = \Big(\frac{\delta u_s}{\delta V}, \frac{\delta v_s}{\delta V}\Big),$$

$$\overline{\delta u}_s = \frac{\delta u_s}{\delta V},\quad \overline{\delta v}_s = \frac{\delta v_s}{\delta V},\quad \bar{\boldsymbol{u}}_s = (\bar{u}_s, \bar{v}_s),\quad \bar{\rho}_l = \frac{\rho_l}{\rho_l^0},\quad \bar{\rho}_s = \frac{\rho_s}{\rho_l^0},\quad \bar{p} = \frac{p}{\rho_l^0 V^2},$$

$$\bar{x} = \frac{x}{L},\qquad \bar{z} = \frac{z}{L},\qquad \bar{t} = \frac{t}{L/V},$$

where L is the length scale, V is the velocity scale, and δV is the particle-velocity lag scale, we nondimensionalize Eq. (4) to obtain

$$\Big(\frac{\partial\bar{u}}{\partial t} + \bar{\boldsymbol{u}}\cdot\nabla\bar{u}\Big) + \frac{\bar{\rho}_s}{\bar{\rho}_l + \bar{\rho}_s}\frac{\delta V}{V}\Big(\frac{\partial\overline{\delta u}_s}{\partial t} + \bar{\boldsymbol{w}}_s\cdot\nabla\bar{u} + \bar{\boldsymbol{u}}_s\cdot\nabla\overline{\delta u}_s\Big) + \frac{1}{\bar{\rho}_l + \bar{\rho}_s}\frac{\partial\bar{p}}{\partial\bar{x}} = \frac{1}{\mathrm{Re}}\Delta\bar{u},$$

$$\Big(\frac{\partial\bar{v}}{\partial t} + \bar{\boldsymbol{u}}\cdot\nabla\bar{v}\Big) + \frac{\bar{\rho}_s}{\bar{\rho}_l + \bar{\rho}_s}\frac{\delta V}{V}\Big(\frac{\partial\overline{\delta v}_s}{\partial t} + \bar{\boldsymbol{w}}_s\cdot\nabla\bar{v} + \bar{\boldsymbol{u}}_s\cdot\nabla\overline{\delta v}_s\Big) + \frac{1}{\bar{\rho}_l + \bar{\rho}_s}\frac{\partial\bar{p}}{\partial\bar{z}} = \frac{1}{\mathrm{Re}}\Delta\bar{v} - \frac{1}{\mathrm{Fr}}. \tag{5}$$

Here $\mathrm{Re} = (\rho_l + \rho_s)VL/\mu$ is the Reynolds number and $\mathrm{Fr} = V^2/(gL)$ is the Froude number. It is seen from Eq. (5) that, in the region with high Reynolds numbers, provided that the inequality

$$\frac{\bar{\rho}_s}{\bar{\rho}_l + \bar{\rho}_s} \frac{\delta V}{V} \ll 1 \tag{6}$$

holds, we can neglect the second terms compared to the first ones in the left side of these equations and solve the system

$$\begin{gathered}\frac{\partial \bar{u}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \bar{u} + \frac{1}{\bar{\rho}_l + \bar{\rho}_s} \frac{\partial \bar{p}}{\partial \bar{x}} = \frac{1}{\mathrm{Re}} \Delta \bar{u}, \\ \frac{\partial \bar{v}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \bar{v} + \frac{1}{\bar{\rho}_l + \bar{\rho}_s} \frac{\partial \bar{p}}{\partial \bar{z}} = \frac{1}{\mathrm{Re}} \Delta \bar{v} - \frac{1}{\mathrm{Fr}}.\end{gathered} \tag{7}$$

Near the solid boundaries, by virtue of the no-slip condition for the liquid, the flow velocity tends to zero together with the inertial terms. At the same time, the particles keep falling under the action of the gravity force. Hence, the corrections to the inertial terms in Eqs. (5) near the solid walls may become comparable with, or even larger than, the first terms because of the presence of terms of the form $\bar{\boldsymbol{w}}_s \cdot \nabla \bar{u}$ and $\bar{\boldsymbol{w}}_s \cdot \nabla \bar{v}$. The same situation may also arise in the case of low-Reynolds-number flows typical of various sedimentation reservoirs and waste-disposal plants.

In the indicated cases, viscous forces start playing the decisive role in liquid motion. Various terms in Eqs. (5) should, therefore, be evaluated in comparison with viscous terms. It is easily seen that the omitted terms in the range of low Reynolds numbers are small if the following inequality is valid:

$$\frac{\bar{\rho}_s}{\bar{\rho}_l + \bar{\rho}_s} \frac{\delta V}{V} \ll \frac{1}{\mathrm{Re}} = \frac{\mu}{(\rho_l + \rho_s)LV}. \tag{8}$$

If inequalities (6) and (8) are both fulfilled, Eqs. (7) can be used everywhere in the flow region. If the length scale in (8) is chosen to be the distance L over which $\mathrm{Re} = 1$ in the vicinity of the solid boundary, then the two inequalities coincide in form. Yet, these inequalities differ in their meaning and are valid for different scales. The parameters V and L in inequality (6) refer to the flow region with high Reynolds numbers, whereas inequality (8) contains scales that characterize the flow region with low Reynolds numbers.

In the case of a low-velocity Stokes flow, the inertial terms in Eqs. (7) can be neglected everywhere in the computational domain; hence, if $\mathrm{Re} \ll 1$, we can confine ourselves to solving the simple system

$$\begin{gathered}\frac{\partial \bar{u}}{\partial t} + \frac{1}{\bar{\rho}_l + \bar{\rho}_s} \frac{\partial \bar{p}}{\partial \bar{x}} = \frac{1}{\mathrm{Re}} \Delta \bar{u}, \\ \frac{\partial \bar{v}}{\partial t} + \frac{1}{\bar{\rho}_l + \bar{\rho}_s} \frac{\partial \bar{p}}{\partial \bar{z}} = \frac{1}{\mathrm{Re}} \Delta \bar{v} - \frac{1}{\mathrm{Fr}}.\end{gathered}$$

To conclude the present consideration, note that system (7) can be used to predict equilibrium (without particle-velocity lag) convection of a two-phase mixture resulting from the different action of the gravity force upon elements having different total local densities $\rho_l + \rho_s$.

Thus, under conditions (6) and (8), free convection of a two-phase mixture consisting of fine particles and an incompressible liquid can be predicted by the following system of equations:

$$\begin{gathered}\operatorname{div} \boldsymbol{u} = 0, \qquad \frac{\partial \rho_s}{\partial t} + \boldsymbol{u}_s \cdot \nabla \rho_s + \rho_s \operatorname{div} \boldsymbol{u}_s = 0, \qquad (\rho_l + \rho_s)\Big(\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla u\Big) + \frac{\partial p}{\partial x} = \mu \Delta u, \\ (\rho_l + \rho_s)\Big(\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \nabla v\Big) + \frac{\partial p}{\partial z} = \mu \Delta v - g(\rho_l + \rho_s), \qquad u_s = u - \tau \frac{du}{dt}, \qquad v_s = v - \tau\Big[\frac{dv}{dt} - g\Big(1 - \frac{\rho_l^0}{\rho_s^0}\Big)\Big].\end{gathered} \tag{9}$$

In the case where inequalities (6) and (8) are fulfilled because of the low mass concentration ρ_s of particles, further simplifications in Eqs. (9) are possible.

System (9) resembles the deep convection equations in terms of the effect of particles on convection of the two-phase mixture [1–4]. Yet, in contrast to the latter equations, this system is substantiated with allowance for inequalities (6) and (8) for arbitrary Reynolds numbers.

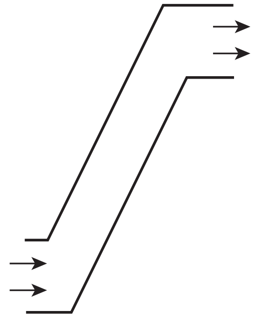

Fig. 1. Schematic representation of the sedimentation cell.

Practical Application of the Model. As an example, we consider sedimentation of coal-slack particles in a shutter sedimentation reservoir [5]. The sedimentation cell is schematically shown in Fig. 1. The cell is shaped as a parallelogram 1 m high and 0.1 m wide; the angle of inclination of the plates is 60°. The cell has an inlet channel 0.1 m high at the bottom on the left side and with an outlet channel of the same size at the top on the right side. Water moves upward along the shutter plates. The particle size ranges from 2 to 60 μm, and the particle concentration varies from 0.2 to 5 kg/m^3. Sedimentation of particles proceeds over the entire height of the shutters; heavy and light particles undergo sedimentation in the bottom part of the cell and in the top part, respectively. According to experimental data, the Reynolds number for the flow around the particles is Re ≈ 1. Hence, the drag of the liquid to the particle flow obeys the Stokes law.

Let us estimate the relaxation time for coal particles in water. In the case of the Stokes drag, this time is expressed by the formula $\tau = d_s^2 \rho_s^0/(18\mu)$. The density of coal is $\rho_s^0 = 1700$ kg/m^3, the particle diameter is $d_s = 5 \cdot 10^{-5}$ m, and the water viscosity is $\mu = 1.42 \cdot 10^{-3}$ Pa$\cdot$sec (at $T = 280$ K). Then, the relaxation time of coal particles is $\tau \approx 1.7 \cdot 10^{-4}$ sec. With due allowance for the smallness of τ, we can omit all terms that contain the relaxation time raised to the second power in formula (2).

Thus, for sedimentation of coal particles in water, the motion of particles smaller than 50 μm in diameter can be assumed stationary, and the particle velocity can be calculated by the formula $\boldsymbol{u}_s = \boldsymbol{u} + \tau \boldsymbol{F} - \tau\, d\boldsymbol{u}/dt$. Note that the acceleration of the carrier liquid in sedimentation reservoirs is normally negligible as compared to the acceleration of gravity. Hence, the particle velocity can be estimated by the formula

$$\boldsymbol{u}_s = \boldsymbol{u} + \boldsymbol{u}_{ss},$$

where $\boldsymbol{u}_{ss} = \tau \boldsymbol{g}(1 - \rho_l^0/\rho_s^0)$ is the equilibrium sedimentation velocity of particles.

Apart from the estimate of the relaxation time, one has to check whether inequalities (6) and (8) are satisfied. In these inequalities, the steady sedimentation velocity $\delta V = u_{ss} = 1.9 \cdot 10^{-4}$ m/sec plays the role of the particle-velocity lag scale. The flow-velocity scale can be estimated from the flow rate of the suspension flow through the treatment plant (0.16 m^3/h). The mean flow velocity is $u = 4.4 \cdot 10^{-5}$ m/sec. Then, the flow Reynolds number between the shutter plates is Re ≈ 3. By performing necessary calculations, we learn that inequalities (6) and (8) result in the relations $1.3 \cdot 10^{-2} \ll 1$ and $0.013 \ll 0.32$, respectively. Thus, for the flow between the shutter plates with the given flow rate of the suspension, inequalities (6) and (8) hold throughout the whole solution domain. The smallness of the relaxation time of particles and validity of the indicated relations enable one to use system (9) for predicting the motion of the suspension of particles in water. Unlike the starting system, system (9) contains no small parameters at the derivatives (is not a "stiff" system), which, in the case of fine particles, allows a considerable reduction of the computing time.

In Cartesian coordinates, system (9), expressed in terms of the variables "vorticity–stream function," which describes sedimentation of fine particles with allowance for the backward influence of particles on the liquid flow, can be written as

$$\frac{d\zeta}{dt} = g\,\frac{1 - \rho_s^0/\rho_l^0}{\rho_l^0}\,\frac{\partial \rho_s}{\partial x} + v\Delta\zeta, \qquad \Delta\psi = \zeta,$$

where $\zeta = \partial u/\partial z - \partial v/\partial x$ is the vorticity and ψ is the stream function given by the laws $\partial\psi/\partial z = u$ and $\partial\psi/\partial x = -v$.

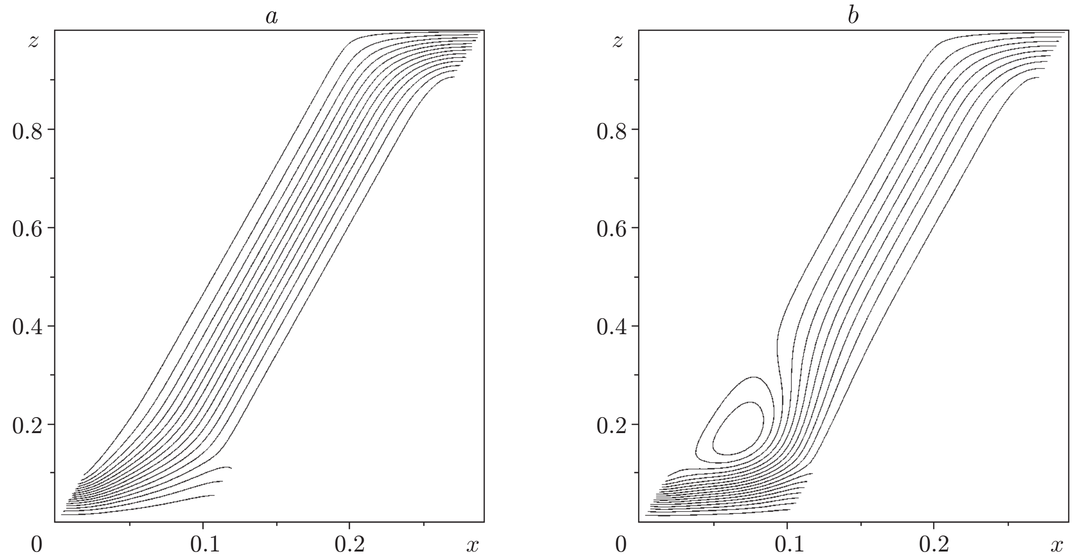

Fig. 2. Streamlines for particle concentrations of 1 (a) and 5 kg/m^3 (b).

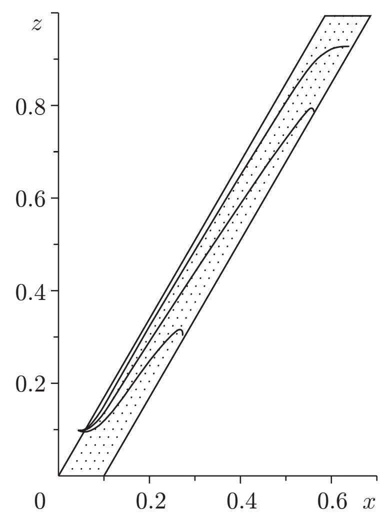

Fig. 3

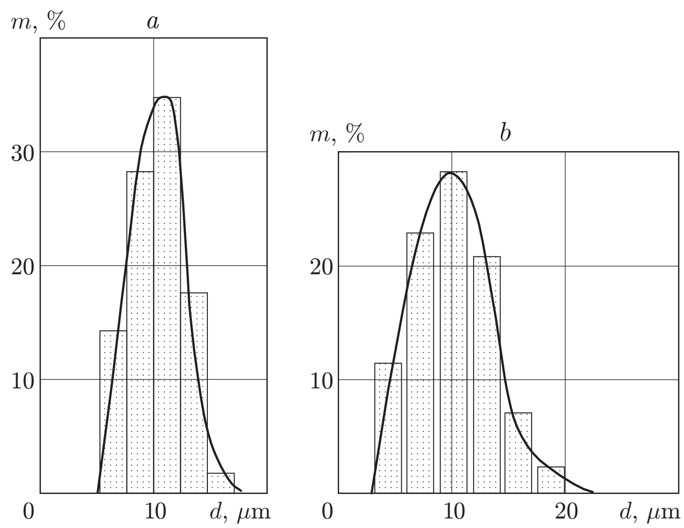

Fig. 4

Fig. 3. Particle trajectories.

Fig. 4. Granulometric composition of deposited control particles: (a) calculations; (b) experiment.

The function ρ_s obeys the transport equation for the substance:

$$\frac{\partial \rho_s}{\partial t} + \frac{\partial (\rho_s u_s)}{\partial x} + \frac{\partial (\rho_s v_s)}{\partial z} = 0.$$

The transport equations for the vortex and for the solid fraction were solved by the MacCormack finite-difference scheme with artificial viscosity introduced into the algorithm [6]. The equation for the stream function was solved by the explicit finite-difference upper-relaxation method. The results were verified by comparing the fluxes of incoming and outgoing particles, with the outgoing flux accessed as the sum of particles deposited in the cell and particles leaving the channel. On the 10×100 computational grid, the difference between the fluxes was within 10%.

Figure 2 shows the streamlines between the sedimentation-reservoir shutters for different particle concentrations. The heavier two-phase mixture tends to spread over the bottom of the sedimentation reservoir. Figure 3 shows the boundary trajectories of particles moving between the shutters. The upper, middle, and lower trajectories refer to 7-μm, 15-μm, and 30-μm particles, respectively. Near the shutter plates, the particles start falling under the action of the gravity force because the velocity of the viscous carrier flow here tends to zero.

The efficiency of sedimentation reservoirs is characterized by the minimum size of trapped particles. Therefore, the primary focus is to be made on the granulometric composition of particles deposited onto the upper part of the shutter plates (10% of their total surface area). The predicted data for particles deposited onto the upper part of the shutter plates are plotted in Fig. 4a. A comparison of the predicted and experimental data (Fig. 4b) demonstrates good qualitative agreement between them.

Conclusions. The mathematical model developed predicts sedimentation of the coarse solid fraction in the liquid rather adequately and can be used to design waste-disposal plants in coal and ore mining and processing and to design various-purpose thin-layer thickeners.

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